THE MATHEMATICS TEACHER

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VOLUME VIII DECEMBER, 1915 NUMBER 2

ROUND TABLE ON THE TEACHING OF CALCULUS.

ELIZABETH B. COWLEY, Vassar College.—A leader of a round-table discussion is not expected to read a formal paper on a minute study of some one part of the topic, but rather to suggest those phases of the subject that will make his talk provocative of discussion. Or, to use a word recently coined, his talk should be "discussative."

As Mr. Smith suggested that it might be well for me to consider the teaching of calculus to girls, let us take that for the first topic. In order to get as quickly as possible to the vital part of the question, suppose we ask: How should a beginning course in the calculus intended for the average young woman in college differ from the course in that subject planned for the average young man in college? If you will permit me to take advantage of my position by replying to this question myself before giving anyone else a chance to do so, I shall answer most emphatically that there should be no difference. It is true that it is possible to find certain problems that might appeal more strongly to some boys than to some girls. But it is possible to find as great differences between different groups of boys. Take, for example, a set of boys whose previous training has all been definitely directed toward preparation for engineering work. Certain problems in calculus that have a very real and vital interest for them will be meaningless to a group of lads in a small rural high school in an agricultural community. Such differences in applications are, after all, superficial. There is a body of fundamental facts and a characteristic mode of

thinking that must be mastered by all. For success in teaching calculus, or any other subject, it is essential that the emphasis be properly placed. Recently we have all heard and read much cheap and superficial tålk about the differences between girls' and boys' work in arithmetic and algebra. These speakers and writers seem to be so overwhelmed by the superficial fact that girls are not boys that they have lost all sense of perspective and have forgotten the important and fundamental fact that girls, as well as boys, are human beings. They are suffering from the same complaint that affected a physicist whom a foreign professor was once trying to describe to an American audience. After telling of the power and insight and industry of this man and how, nevertheless, he had never quite arrived at the goal, the professor said "The trouble with him was that he always got his empha'sis in the wrong places."

Let us turn, now, to some other questions that may prove of interest. When should a first course in calculus be introduced? There has been a tendency in colleges in recent times to push the elementary calculus back into the sophomore or freshman year. This is due, usually, to pressure from the allied departments, which need the elements of differentiation and integration by the end of the sophomore year, at least. Another question which this one suggests is whether it is possible to have any calculus in the secondary schools. If so, shall it be a formal course in calculus, or merely a placing of emphasis in the algebra upon the ideas of function, graph, etc.? The question of calculus in the secondary schools has received much attention in Europe.

Another point toward which discussion might profitably be directed is the relative emphasis to be placed, in a beginning course, upon the purely formal side and the applications.

Again, there may be considerable difference of opinion as regards the best time for introducing integration. For instance, is it wise to take up the simplest algebraic integrals before the study of the differentiation of the trigonometric functions?

The last topic which I shall suggest is the importance of laying sufficient emphasis upon the simplest underlying principles of the calculus. Too often students who can readily differentiate and integrate quite complicated forms have the humiliat-

ing (but stimulating) experience outside the class-room of being asked what the calculus really is and being unable to give any coherent and satisfactory answer. Of course, this is a deep question for a young student. But, is it not possible, even in a first course in the calculus, for the student to gain a clear idea of the simplest fundamental principles? By so doing, he will greatly increase the pleasure and profit to be derived from his work.

W. H. METZLER, Syracuse University.—In considering the teaching of calculus to students in a college of liberal arts this speaker emphasized the importance of students having a very thorough knowledge of the derivatives and integrals of all fundamental forms. Integration is an art as well as a science and students should be taught the short and concise way of integrating any given expression. To this end it is poor training to use, as illustrations of any given method of integration, examples which would better be worked by some other methods. Students should not use a table of integrals but should learn how to integrate the type forms.

WILLIAM J. BERRY, Brooklyn Polytechnic Institute.—My colleagues, I know, will excuse me if I confine my remarks about the teaching of calculus entirely to the question of the instruction of engineering students. All my professional life has been spent in engineering schools, and I prefer to speak only of those things with which I am acquainted. There are two problems to the consideration of which I invite your attention for a few moments. First, what shall be the attitude of the teacher of calculus toward his subject? and second, what is the general attitude of the student toward the calculus?

In an article published in the *Engineering Record* for February 27, 1915, Professor Swain, of Harvard University, says: "I think that the fact that engineering is to so large an extent a mathematical subject is one of the main reasons why the engineer is not recognized. In my opinion, there is scarcely anything that tends more to narrowness of view than dealing all the time with problems that can be solved only by rigid mathematical processes, because the problems which confront us are

not mathematical, the data are shifting, variable, and the human element comes in."

Within recent years there has arisen what is sometimes referred to as the Perry method of teaching Calculus, and in my opinion there may be some connection between this method and the attitude which it represents, and the criticism which I have just quoted. I am quite sure that all of us, as teachers of mathematics, dissent most strongly from the point of view expressed by Professor Swain. It is gratifying to find that he does not voice the sentiments of all the members of his profession. In the *Record* for March 20, 1915, Mr. Allen Hazen, commenting upon Professor Swain's article, says: "To the writer one of the most surprising things in Professor Swain's statement is his reference to the narrowing influence of mathematics. The writer has been inclined to think that the study of mathematics is useful in tending to clear and logical thinking and accurate expression, and he has been disposed to think that the engineering schools might well require more mathematics rather than less. The fact that the practical problems of life are not mathematical ones, but are very much more complex, is another matter, and one that the man must adjust himself to when he encounters the problem of life."

And a little later Mr. L. C. Fritch says: "There is no better training for the mind than a course in mathematics, and in this respect the training of the engineer is superior to that of the lawyer or business man. Mathematics is more truly an exact science than the law or any other of the arts."

Although we do not agree with Professor Swain, yet his criticism, coming from so eminent an authority, must necessarily give us pause, and we should inquire carefully into the subject and endeavor to find out to what extent such strictures are justified. The teacher may take either of two attitudes with regard to the presentation of the calculus to students of engineering. He may consider it as a tool, of which they must become the masters, which they must be able to use with facility and accuracy, and he may rest content with that. Or he may regard it as a science in which they are to be instructed, which is not only to teach them facts, but to develop their power of thought, to stimulate their imagination, and to give them the

vision without which neither mathematician nor engineer can be truly great. He is concerned rather with the fundamental principles and methods than with applications; his students are to be thinkers and not merely differentiating or integrating machines. The Perry method takes rather the first of these two points of view, and it is my opinion that taught in this fashion the calculus may well be occupying more of the time of the engineering student than is justified, and may lead to precisely that type of narrowness of which Professor Swain speaks. If, on the other hand, the teacher regards his work as the interpretation of one of the great branches of mathematical science, if he feels that his mission is not to cram the mind of his student with facts and formulas, to prepare them to compute radii of gyration and moments of inertia for engineering handbooks, but to stimulate their imaginations, to clarify their ideas, and to give them power of thought and accuracy of expression,—then I believe that the time allotted to the calculus is rather too short than too long.

When we come to a consideration of just what topics should be included in a first course, what methods of proof should be used, how far mathematical rigor should be insisted upon, we find that the questions which present themselves are not at all easy to answer. The most satisfactory course that has yet been outlined is probably the one prepared by the committee of the Society for the Promotion of Engineering Education and published in its syllabus. Even then the teacher will find that there is more than can be adequately presented in a course of a hundred hours. Personally, I believe that the time allotted to the calculus in a school of engineering should be about fifty per cent. more than this.

Mathematics is a science which above all others should be honest and straightforward. I think it a grave mistake to present to the student as a proof an argument which we ourselves know to have weak points in it. You may say, and rightly, that it is not possible to give the student in his second year in college rigorous proofs of the various theorems needed in the calculus. That I cheerfully grant, but I think in fairness to him, and as a matter of common honesty, we should point out the deficiencies, assure him that the rigor that is lacking can

be supplied, and explain to him that the reason that it cannot be done then and there is because of his own inadequate preparation and insufficient background. My personal experience has been that the student does not doubt the truth of the theorem when you tell him that what you present is not a proof, but is merely making probable the theorem through geometric intuition; and I know, too, that if in more advanced work he finds that the demonstrations that have been given him are faulty, he begins to suspect not only all the demonstrations with which he is familiar, but the accuracy and truth of the theorems as well.

I believe that the justification for the inclusion in an engineering course of a course in the calculus is due not so much to the fact that it will fit the student more quickly for the earning of a living wage, as to the far more important fact that it is the basis of all future work in analysis, that it lies at the foundation of many of the most important branches of engineering, that progress in the development of the science (and this is particularly true of electrical engineering) is conditioned largely by familiarity with mathematical processes and facility in their use. As an example of this, I need only to cite the very remarkable work of Professor Kennelley, of Harvard University, in the application of functions of complex variables to problems of alternating-current electricity.

And now what about the attitude of the student? We are all quite familiar, I am sure, with the fact that the calculus is usually the best hated course in the institution, that in more than one school those who have successfully passed their final examination in the subject join in a solemn *auto da fe* in which the offending text-books are formally committed to the flames. Personally, I never have been able to see why this should be so. I do not believe that there are any great difficulties inherent in the subject, and am constrained to the opinion that the trouble is either with the method of presentation or with the inadequate preparation of the students themselves. I should be very loath to believe that the former explanation is the true one, for the attitude of the student seems to be pretty much the same even when the teachers are men of whose ability as mathematicians and as instructors, we can have very little doubt.

For some years I have been studying this subject very care-

fully in my own classes and have found that in perhaps nine cases out of ten when a student failed to understand a demonstration it was because the algebra, the trigonometry, or the analytic geometry employed had exceeded his knowledge of the subjects; that when a student failed to get a problem the trouble was more apt to be with the manipulation of the quantities involved than it was with the fundamental principle of the calculus underlying it. The new ideas introduced into the calculus are comparatively few, and with careful explanation ought to give no unusual amount of trouble. If we examine the average text-book we shall find that it consists very largely of applications of these few principles. In my own institution we have succeeded to a very satisfactory degree in eliminating much of this feeling against the calculus, and it has been accomplished by calling the attention of the students, publicly and privately, not in a general way but very specifically, to the fact that their difficulties are due, not to the subject that they are at present studying, but to the subjects with which they had fondly supposed they were finished and which they thought were forever put behind them.

Let me say in conclusion that if the teacher of the calculus in an engineering school will steadfastly refuse to lower his ideals, to turn what is a wonderful science into a mere tool, if he will endeavor to turn out men who have power to think and to analyze rather than mere perambulating integraphs, if he will remind the student wherever possible that the trouble is not with the calculus but with himself and his lack of preparation, he will find not only that the results are more satisfactory, but that the work itself will bring that interest and that satisfaction which are the teacher's true reward.

CHARLES C. GROVE, Columbia University.—It was of interest to hear Professor Metzler's experience and practice with regard to tables of integrals. I have not ceased to rejoice that I was not brought up to use tables as a student. I feel that if the tables are introduced before the student has mastered the casting of integrals into standard forms and integrating them by observation, before he has learned to use rather freely the special method of integration, and the special transformations

that make integrals readily integrable, he comes to rely on the tables and will likely never become proficient in the art of integrating.

Where a full year is devoted to the first course in the calculus, I prefer to make no mention of tables during the first half-year. For example, I would cover the first 200 pages of Davis or 133 pages of Osgood before giving tables and would and do give more methods and substitutions of integration than given by either of these authors.

Further, many integrands are not given in a form that permits the use of tables directly. The student needs to learn how to recast these integrands and by the time he has done that the integral is very often in a form that may be integrated by observation, if he has learned twenty to twenty-five standard forms through using them frequently. For example, of the 164 integrals given in Osgood, pp. 446–451, almost all are more easily and quickly integrated without reference to Peirce's Short Table of Integrals than by trying to cast them into a form to which the table may be applied.

The privilege of using tables early may readily become a snare and a delusion. Later, after the student has learned how to integrate without resort to tables, for rapid work in practice the tables may well be used and have their proper place.